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IDENTIFICATION OF THERMAL CHARACTERISTICS OF BUILDING MATERIALS AT HIGH TEMPERATURES

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ABSTRACT

Identification of thermal characteristics of building materials under hard experimental conditions cannot be done using standard laboratory techniques. However, the data from non-stationary hot-wire (or similar) experiments, supported by recent European standards, are expected to give reasonable effective values of (at least) heat conductivity and heat capacity under the assumption of validity of a (semi)linear macroscopic heat transfer equation with initial and boundary conditions. This paper demonstrates i) how some results of this type can be obtained thanks to the proper study of analytical solutions with exponential integrals (hidden in standards), ii) how such approach can be improved using the system of Bessel functions, iii) what difficulties must be overcome in the general case, applying the least squares and conjugate gradient techniques to both deterministic and stochastic problems.

Keywords: building materials, thermal characteristics at high temperatures, inverse problems.

INTRODUCTION

Standard methods of identification of basic thermal characteristics of building materials, for macroscopically homogeneous and isotropic material structures, namely of their effective thermal conductivity λ [W/(K·m)] (crucial for the thermal insulation properties) and heat capacity c [J/(K·kg)] (important in the analysis of thermal accumulation), as described in (Černý, 2010), p. 105 (for separate steady-state measurements of λ and various calorimetric ones for c), are not applicable under hard experimental conditions. In particular, this is true i) for fire-clay bricks at high temperature, ii) during early-age changes in maturing concrete mixtures, iii) for heat storage equipments in systems of effective exploitation of sun energy using optical fibres, iv) for advanced phase-change insulation materials – cf. (Šťastník, 2010). Similar arguments are valid also for the heat transfer coefficient γ [W/(K·m^2)] between various materials. A more promising way of direct non-stationary measurements of the temperature T and the controlled heat fluxes q, combined with non-trivial computational techniques, compatible with (Bochev, 2009), p. 49 (with quasi-solutions in sense if (Isakov, 2006), p. 32), refers to various modifications of the least squares formulation of an original evolutionary variational problem. It can be traced from and (Duda, 2003) and (Achtolungo, 2008); available alternatives to the least squares approach are discussed in (Colaço, 2006).

Real non-stationary measurement devices consist usually from several material layers, including an analyzed specimen, whose material parameters are unknown in advance. Some heat fluxes are forced by electrical heating; due to the measurement configurations, the most frequently used approaches are the hot-plate method by (Klarsfeld, 1992), the hot-wire method by (André, 2003) and the hot-ball method by (Kubičár, 2007). In addition to the controlled heat source, some temperature sensors must be included in the measurement system; if both the temperature and the heat fluxes can be evaluated at a substantial part of the boundary then the identification of λ, c and γ is possible, thanks to the overdetermined boundary conditions. Various modifications of the basic configurations are available, as certain additional insulation layers, the second (cold) plate, etc. – cf. (Šťastník, 2007). Most measurement devices try to arrange a simple (nearly) closed physical system, although the real (at least) thermo-, hygro- and chemo-mechanical material behaviour is typically a result of interaction of several physical and chemical processes, whose proper analysis requires a multi-scale approach to the balance laws of classical thermodynamics – cf. (Vala, 2011). Since T may be not continuous on the boundary of a specimen, the (a priori unknown) interface heat transfer coefficient γ, in general variable on the boundary (but not in time), can be taken into account, similarly to the Robin coefficient in (Jin, 2008).

The form of mathematical description is derived from the experimental arrangement. Whereas the hot-plate experiments are simulated using the Cartesian coordinates (typically under additional symmetry assumptions), the hot-wire experiments apply the cylindrical and the hot-ball experiments the spherical ones. In some cases this access offers a possibility to exploit special properties of (semi)analytical solutions with trigonometric, Bessel, etc. functions, combined (for direct problems) with the general Fourier method. Following such classification, we shall pay attention namely to a model case of the hot-wire measurements system and related computational tools.

The approach of (EN ISO, 2010), exploiting the classical analysis of (Carslaw, 1946), considers λ and c as constants (at least for the temperature range of separated measurements). Moreover, it ignores the third, fourth, etc. additive terms in the power series expansion of exponential integrals, thus it should be natural to remove this formal simplification. Just this *first step* is able to get a realistic estimate of c, in addition to the improved value of λ, using the standard Newton iteration procedure together with the least squares optimization technique for two real variables to handle uncertain data.

The *second step* relies on the removal of further non-physical assumptions, as of the zero wire thickness and of the infinite specimen size. This leads to the analytical solutions based on the theory of Bessel functions and to much more complicated optimization algorithm, whose derivation needs formal MAPLE manipulations, but still with two real variables. From the macroscopic point of view, we consider an isotropic material, thus one scalar characteristic λ is be sufficient; to remove this assumption, it is not difficult to take heat fluxes $(\lambda_1 \partial T/\partial x_1, \lambda_2 \partial T/\partial x_2, \lambda_3 \partial T/\partial x_2)$, containing a triple of (a priori unknown) factors $(\lambda_1, \lambda_2, \lambda_3)$, instead of heat fluxes $\lambda \nabla T$ just with λ only. Another simplification, following (Wullschleger, 2008), consists in the (seemingly too strong) assumption that λ, c (and also γ, needed in further considerations only – a perfect interface with continuous T is still assumed here) are independent of T; however, all local forced non-stationary temperature changes near the measurement device are negligible in comparison to the ambient temperature in view. The original software package, applicable also (for comparison, as the limit case) to simplified computations, as well as all measurement equipments, have been created in the Laboratory of Building Physics at Brno University of Technology (BUT). The received val-

ues of λ and c usually seem to be rather realistic, but non-negligible differences between experimental data and simulation results, not explainable by statistical arguments, occur also here.

The numerical simulation of the above mentioned heating experiments with some estimated values of λ, c (and γ, if considered in the model) starts an iteration procedure, whose aim is to obtain the optimal values of λ, c (and γ) in a reasonable least squares (or similar) sense, working with the time-variable difference between the calculated and recorded temperature. An alternative approach, studying the difference between the predicted and controlled boundary heat flux, is rarely used because of the computational difficulties with unstable boundary conditions. Instead of c, $\kappa = \rho c$ [J/(K.m^3)], using the material density ρ [kg/m^3], is often considered; such formal transformation is acceptable because the experimental setting of ρ is usually much more simple than that of the remaining characteristics. Another material characteristic, frequently used in the literature, is the thermal diffusivity α [m^2/s], introduced as $\alpha := \lambda/\kappa$. In the hypothetical case of pure heat conduction and quite exact measurements, computations of the optimal values of λ, κ (and γ) correspond to a zero least squares sum, otherwise such sum can serve as a basis for uncertainty considerations, e. g. for the Sobol sensitivity analysis by (Kala, 2011).

The most general *third step* requires proper numerical analysis both in the 3-dimensional Euclidean space (e.g. using the finite/infinite element technique) and at the simulated time interval (using the Rothe sequences or the Galerkin approach) in the deterministic case, but (to handle the data uncertainty) also the stochastic modifications of such methods like (Zabaras, 2004), whose existence and convergence theory have not been closed yet. The optimization procedure, coming from the least squares approach, working with material parameters from Lebesgue and Sobolev spaces, can be decomposed to particular steps related to direct, sensitivity and adjoint problems, using the conjugate gradient method with special regularization terms. However, the need of such robust and efficient solver of practical inverse problems remains as a motivation for future research.

Fig. 1 Practical implementation of the hot-wire method, following (EN ISO, 2010).

IDENTIFICATION PROCEDURE BASED ON EXPONENTIAL INTEGRALS

The European standard (EN ISO, 2010) contains an explicit (seemingly strange) logarithmic formula for the evaluation of λ, supplied (for uncertain measurements) by the least-squares (linear regression) approach to data fitting. However, as shown in (Bilek, 2006), it can be identified with the fundamental solution of the heat conduction equation, satisfying the realistic boundary conditions in certain limit sense, well-known from (Carslaw, 1946), where in the additive decomposition of an exponential all terms except the first two are removed; this can be justified by the location of temperature sensors close to the heating wire. Such approach enables us to calculate (approximately) λ without the a priori knowledge of α; unfortunately, no information referring to κ is then available (because it was hidden in the removed terms of higher orders containing α). We shall demonstrate that the proper analysis of the above sketched problems offers a possibility to identify both λ and κ from the same data set. Moreover, we shall show later how some unpleasant physical and geometrical assumptions can be modified to be more realistic, using the properties of Bessel functions by (Culham, 2004), instead of the old analytical results from (Carslaw, 1946).

Following (EN ISO, 2010), let us assume that some constant heat Q [W/m], starting from the zero initial time, is generated per unit length of a very long and thin wire, located in the axis of the circular cylinder with a very large radius, occupied by the material specimen. Let $T(r,t)$ be the temperature field defined for any positive radius r (a distance from the axis of rotation) and each positive time t (for some measurement period in practice) and T_0 the constant temperature of the surrounding environment. Then, by (Bilek, 2006), referring to (Carslaw, 1946), or (Borukhov, 2006), applying the notation $\beta_0 := Q/(4\pi\lambda)$, $\beta := 1/(4\alpha)$, we have

$$T = \beta_0 \, \mathrm{Ei}(\beta r^2/t) + T_0 \qquad \text{with} \quad \mathrm{Ei}(.) := \int_{.}^{\infty} \frac{\exp(-u)}{u} \mathrm{d}u \, . \tag{1}$$

Indeed, using dot symbols for partial derivatives with respect to t and prime symbols for those with respect to r, it is easy to verify that T from (1) satisfies the classical Fourier equation of heat conduction (without internal heat sources) with constant characteristics λ and κ in polar coordinates

$$\kappa \dot{T} + \frac{\lambda}{r}(rT')' = 0 \tag{2}$$

together with the obvious initial condition $T(.,0) = T_0$ and the with couple of boundary conditions

$$\lim_{r\to\infty} T(r,.) = 0, \qquad \lim_{r\to 0+} \frac{-\lambda T'(r,.)}{Q/(2\pi r)} = 1 \tag{3}$$

where the first limit guarantees the absence of heat fluxes from external environment and both the numerator and the denominator in the second limit represent the heat flux [W/m^2] on the surface of cylinder with a fixed small radius (this is just the announced way how to avoid the realistic finite radius and all material characteristics of a wire). Clearly the data for $t = 0$ (and also $t \to 0$ in practice), thanks to the discontinuity of heat generated into the system (forcing the application of Dirac measures and Heaviside functions in (Carslaw, 1946)), are then not employable in any credible identification procedure for λ and κ, in particular for λ and α from (1); for the special case of the simplified evaluation of λ this observation is reflected by (EN ISO, 2010), too.

Let us assume that all sensors recording the temperature are located at $r=\delta$ where a distance δ must be a very small positive number by (EN ISO, 2010) (the measurement could be performed as close as possible to the wire surface), but is allowed to be finite in our considerations. Let m be a number of measurement time steps; the initial time $t=0$ is not included. Let $t_1,...,t_m$ ($0<t_1<...<t_m$) be discrete measurement times and $T_1,...,T_m$ corresponding temperature values at $r=\delta$. All differences T_s-T_{s-1} with $s\in\{2,..,m\}$ should correspond to the experimental temperature differences τ_s. For simplicity, only one recorded temperature value is considered in every discrete time here; the generalization over all available data is obvious. Thus, using the notation $\beta_1:=\beta\delta^2$, we have to minimize a function

$$\Phi=(1/2)\sum_{s=2}^{m}(\tau_s-(T_s-T_{s-1}))^2 \tag{4}$$

of two positive variables β_0 and β_1 (transformed from λ and α easily).

Let $\Phi_{,i}$ and $\Phi_{,ij}$ denote the derivatives $\partial\Phi/\partial\beta_i$ and $\partial^2\Phi/\partial\beta_i\partial\beta_j$ with $i,j\in\{0,1\}$. For $\beta_{1s}:=\mathrm{Ei}(\beta_1/t_s)$, $\beta_{1s}^*=\exp(-\beta_1/t_s)-\exp(-\beta_1/t_{s-1})$ and $\varepsilon_s=\beta_0\beta_{1s}-\tau_s$ with $s\in\{2,..,m\}$ we receive the explicit formulae (the MAPLE support is welcome)

$$\Phi=(1/2)\sum_{s=2}^{m}\varepsilon_s^2\ ,\qquad \Phi_{,0}=\sum_{s=2}^{m}\varepsilon_s\beta_{1s}\ ,\qquad \Phi_{,1}=-(\beta_0/\beta_1)\sum_{s=2}^{m}\varepsilon_s\beta_{1s}^*\ ,$$

$$\Phi_{,00}=\sum_{s=2}^{m}\beta_{1s}^2\ ,\qquad \Phi_{,01}=-(1/\beta_1)\sum_{s=2}^{m}(2\beta_0\beta_{1s}-\tau_s)\beta_{1s}^*\ ,$$

$$\Phi_{,11}=(\beta_0/\beta_1)^2\sum_{s=2}^{m}\beta_{1s}^{*\,2}+(\beta_0/\beta_1)\sum_{s=2}^{m}\varepsilon_s\left(\mathrm{Ei}(\beta_1/t_s)/t_s-\mathrm{Ei}(\beta_1/t_{s-1})/t_{s-1}\right)(\beta_0/\beta_1^2)\sum_{s=2}^{m}\varepsilon_s\beta_{1s}^*\ .$$

Clearly we need $\Phi_{,0}=\Phi_{,1}=0$. Taking (for sufficiently small δ) $\beta_1\approx 0$ together with $\mathrm{Ei}(.)\approx -C_e-\ln(.)$ (the Euler-Mascheroni constant $C_e\approx 0.5772156649$ is not needed in numerical calculations), for $\gamma_s:=\ln(t_s/t_{s-1})$ with $s\in\{2,..,m\}$ we obtain the very simple formula

$$\beta_0\approx\sum_{s=2}^{m}\gamma_s\tau_s/\sum_{s=2}^{m}\gamma_s^2, \tag{5}$$

which is identical with that for the identification of λ from (EN ISO, 2010). More generally, we are allowed to choose β_0 from (5) as the first estimate together with

$$\beta_1\approx\sum_{s=2}^{m}(1/t_s-1/t_{s-1})(\tau_s/\beta_0-\gamma_s)/\sum_{s=2}^{m}(1/t_s-1/t_{s-1})^2$$

and apply the Newton iteration procedure

$$\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}\leftarrow\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}-\begin{bmatrix}\Phi_{,00} & \Phi_{,01}\\ \Phi_{,01} & \Phi_{,11}\end{bmatrix}^{-1}\begin{bmatrix}\Phi_{,0}\\ \Phi_{,1}\end{bmatrix}\ ,$$

i.e., following the Cramer rule, simply $\beta_0\leftarrow\beta_0-B_0/B$ and $\beta_1\leftarrow\beta_1-B_1/B$ where

$$B:=\Phi_{,00}\Phi_{,11}-\Phi_{,01}^2\ ,\quad B_0:=\Phi_{,0}\Phi_{,11}-\Phi_{,1}\Phi_{,01}\ ,\quad B_1:=\Phi_{,1}\Phi_{,00}-\Phi_{,0}\Phi_{,01}\ .$$

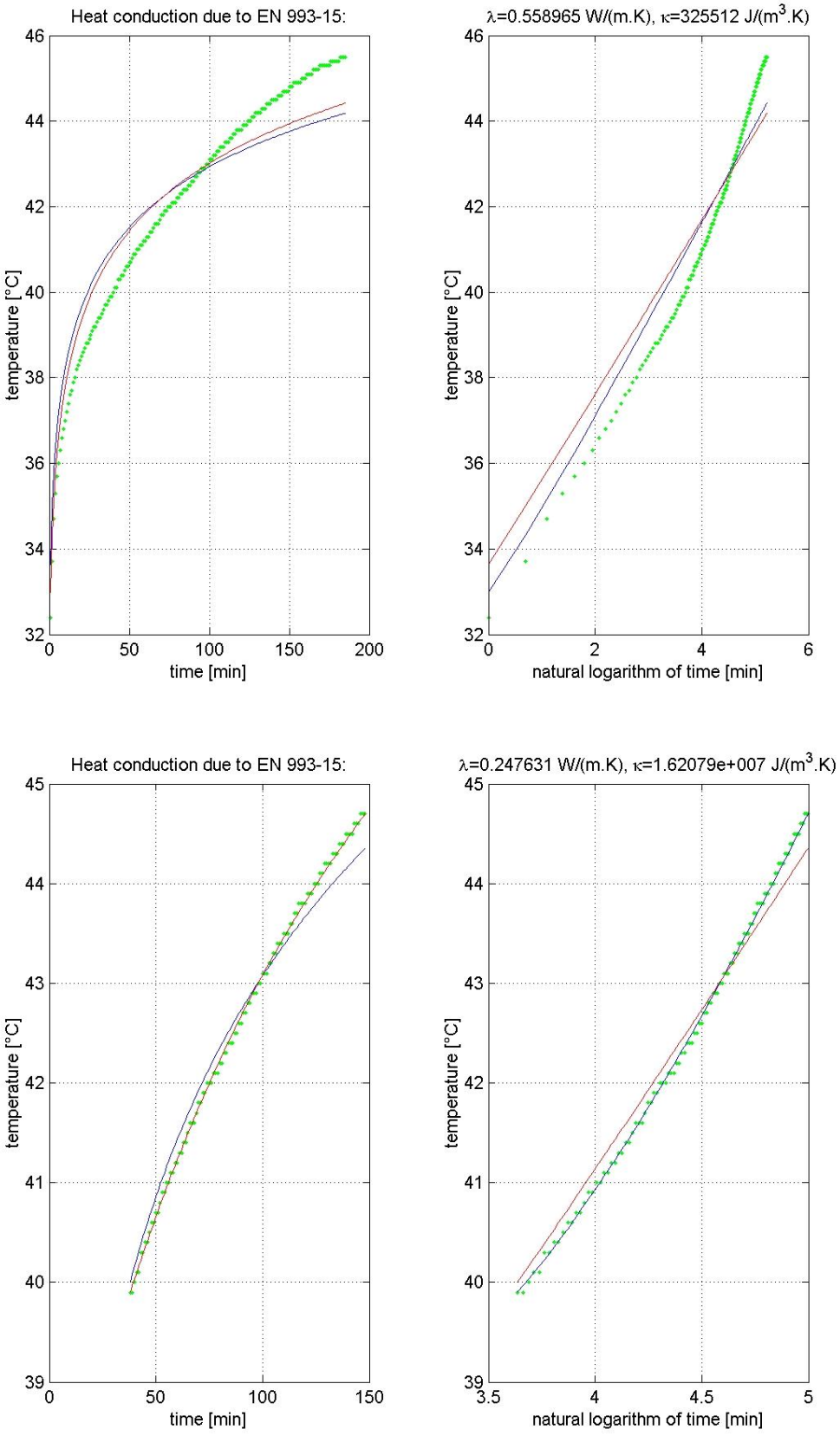

Fig. 2 Development of temperature in time – complete data set (upper graph) and reduced data set (lower graph).

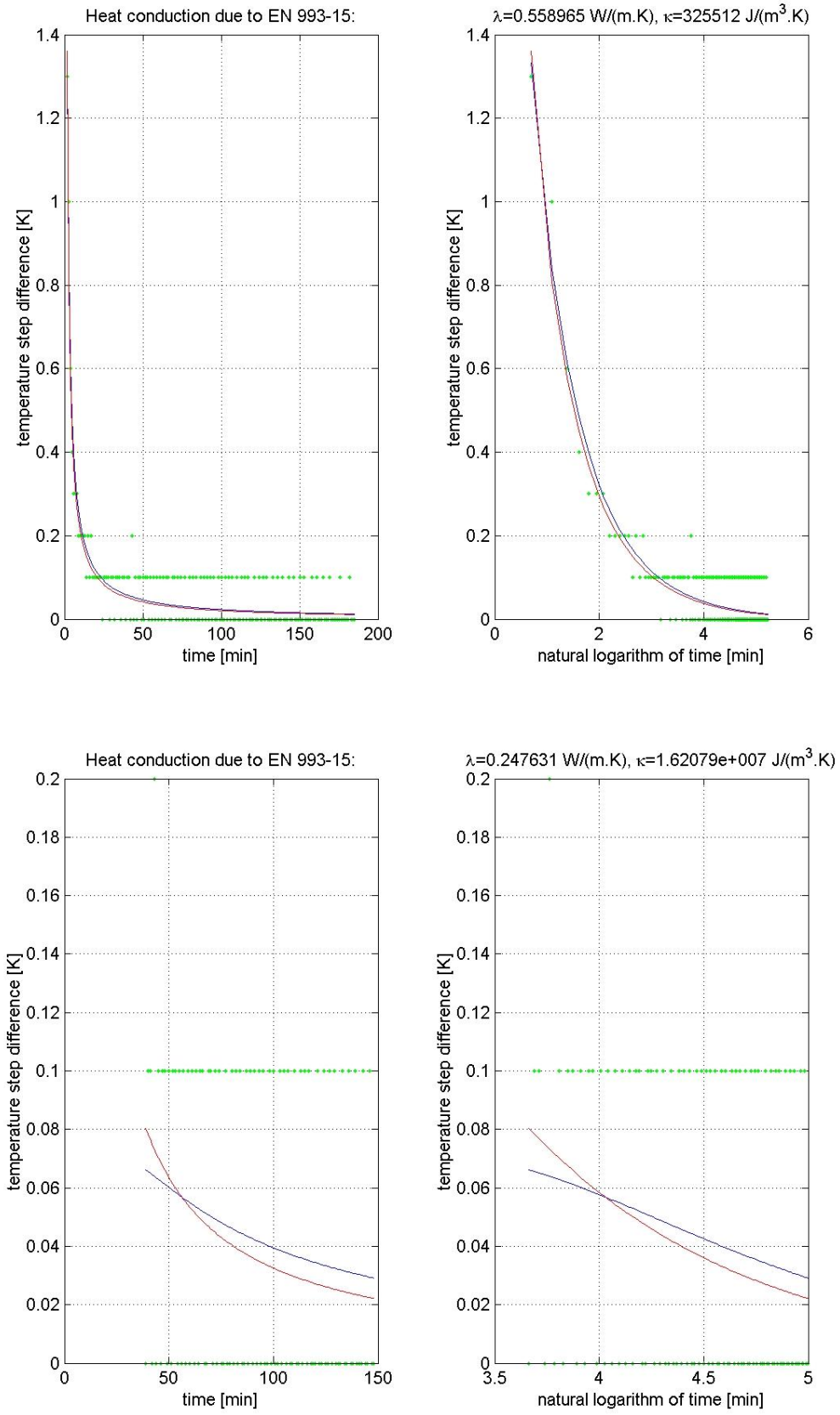

Fig. 3 Development of difference of neighbour temperatures in time – complete data set (upper graph) and reduced data set (lower graph).

This enables us to determine (at least theoretically, with a higher accuracy) both β_0 and β_1, consequently also λ, α and κ.

As an illustrative (and cautionary) example, let us present the hot-wire measurement equipment by Fig. 1, described in (Šťastník, 2012). Just one temperature value was recorded in each time step; the sensitivity of its measurement was only 0.1 K. Fig. 2 and Fig. 3 present the results of the above sketched identification procedure, including the final values of λ and κ; the logarithmic-scaled time is presented alternatively because of the logarithms hidden in (5). Fig. 2 shows the temperature development in time and documents the computational algorithm for all available temperature measurements: the green dots represent the recorded temperature values, the dashed red line the estimate by (EN ISO, 2010), coming from the approximation (5), the full blue line the final result of the proper minimization of (4). Since (EN ISO, 2010) recommends certain data reduction, both identification procedures have been done for complete and reduced data separately – with quite other results. Fig. 3, corresponding to Fig. 2, displaying differences of neighbour temperatures in time for both mentioned cases, discovers the most probable source of this fault: the low sensitivity of recorded temperature values. Clearly some improvement of this method is needed. One could rely on better results for a higher distance of temperature sensor(s) from the hot wire, but this i) requires substantial reconstruction of the measurement device, ii) looses the compatibility with (EN ISO, 2010).

IDENTIFICATION PROCEDURE BASED ON BESSEL FUNCTIONS

The generalization of the above sketched approach, removing mathematical and physical simplifications, can be done in more directions. However, being motivated from the results of MATLAB-supported practical calculations with experimental data, we shall try to replace the rather artificial boundary conditions (3), following (Singh, 2010), by more realistic ones.

Let a be the outer radius of a specimen and $\delta < a$ a wire radius. Let us introduce the brief notation for scalar products in the special Lebesgue weighted spaces (cf. (Fučík, 1980))

$$(\phi,\phi)_r = \int_0^a \phi(r) r \phi(r) \mathrm{d}r \qquad \text{for all } \phi,\phi \in L_r^2(0,a) \ ,$$

$$(\phi,\phi)_{r0} = \int_0^\delta \phi(r) r \phi(r) \mathrm{d}r \qquad \text{for all } \phi,\phi \in L_r^2(0,\delta) \ ,$$

$$(\phi,\phi)_{r1} = \int_\delta^a \phi(r) r \phi(r) \mathrm{d}r \qquad \text{for all } \phi,\phi \in L_r^2(\delta,a) \ .$$

Material characteristics λ,κ,α will be taken as simple functions of r, with values equal to a priori known constants $\lambda_0,\kappa_0,\alpha_0$ for $0 \le r \le \delta$ and unknown ones $\lambda_1,\kappa_1,\alpha_1$ for $\delta \le r \le a$ (although their rather good estimates may be available by the previous section); moreover we shall need $\lambda_* := \lambda_1/\lambda_0$, $\kappa_* := \kappa_1/\kappa_0$ and $\alpha_* := \alpha_1/\alpha_0$.

Let V be the space of admissible test functions, i. e., applying the notation of special Sobolev weighted spaces (cf. (Fučík, 1980)) again, the space of all $v \in W_r^{1,2}(0,a)$ such that

i) $v(r) = v_0(r)$ for $0 \le r \le \delta$ and some $v_0 \in W_r^{1,2}(0,\delta)$,

ii) $v(r) = v_1(r)$ for $\delta \le r \le a$ and some $v_1 \in W_r^{1,2}(\delta,a)$ satisfying $v_1(a) = 0$.

Let H be the space introduced in the same way as V except L_r^2 inserted instead of $W_r^{1,2}$ everywhere. Using such notation, we are able to convert (2) into the form

$$(v,\kappa\dot{T})_r = (v,\lambda(rT')'/r)_r + (v,g)_r \tag{6}$$

where $g := Q/(\pi\delta^2)$ for $0 \le r \le \delta$ (any better information on the distribution of g in a wire is usually missing), zero otherwise. For positive times t we have to find $T(.,t) - T_0$ from V with $\dot{T}(.,t)$ from H.

Let us consider the decomposition $T(r,t) = T_\sigma(r) + \theta(r,t)$ where

$$T(r,t) = T_\sigma(r) + \theta(r,t) \qquad \text{with} \quad \theta(r,t) = \sum_{i=1}^{\infty} \varphi_i(r)\psi_i(t) \ ; \tag{7}$$

the corresponding initial conditions are $T(.,0) = T_0$ and $\theta(.,0) = T_0 - T_\sigma(.)$ and the boundary (including the internal interface) ones are

$$\begin{aligned} &T'(0,.) = 0 \ , \quad \lambda_0 T'(\delta_-,.) = \lambda_1 T'(\delta_+,.) \ , \quad T(a,.) = 0 \ , \\ &\theta'(0,.) = 0 \ , \quad \lambda_0 \theta'(\delta_-,0) = \lambda_1 \theta'(\delta_+,0) \ , \quad \theta(a,.) = 0 \ , \\ &\lambda_0 T_\sigma(\delta_-) = \lambda_1 T_\sigma(\delta_+) \ , \quad T_\sigma(a) = T_0 \ . \end{aligned} \tag{8}$$

($_+$ and $_-$ signs as indices after δ refer to left and right limits). Here T_σ can be derived as an analytical solution for the stationary case (with zero κ formally)

$$T_\sigma(r) = \begin{cases} Q/(2\pi\lambda_1)\ln(a/\delta) + Q/(4\pi\lambda_0)(1-(r/\delta)^2) & \text{for } 0 \le r \le \delta \ , \\ Q/(2\pi\lambda_1)\ln(a/r) & \text{for } \delta \le r \le a \ . \end{cases}$$

Utilizing the properties of Bessel functions

$$J_n(r) = \frac{1}{\pi}\int_0^\pi \cos(r\sin\xi - n\xi)\mathrm{d}\xi \qquad \text{with } n \in \{0,1,2,...\} \ ,$$

namely $J_0(r) = -J_1(r)$, $J_1(r) = J_0(r) - J_1(r)/r$, etc., by (Culham, 2004), Chap. 7-8 (with the title *Bessel functions of the first and second kind*), we can see that

$$r^{-1}(rJ_0(\omega r))' + \omega^2 J_0(\omega r) = 0 \tag{9}$$

for any real ω, it is natural to find the zero points of Bessel functions, i. e. to solve equations $J_0(\omega_i a/\sqrt{\alpha_*}) = 0$ for unknown parameters ω_i with $i \in \{1,2,...\}$, and to choose

$$\varphi_i(r) = \begin{cases} \beta_i J_0(\gamma_i\omega_i r) & \text{for} 0 < r < \delta \ , \\ J_0(\omega_i r/\sqrt{\alpha_*}) & \text{for} \delta < r < a \ , \end{cases} \tag{10}$$

to satisfy boundary conditions $\varphi_i(0) = 0$, $\varphi_i(a) = 0$ automatically and

$$\varphi_i(\delta_-) = \varphi_i(\delta_+), \qquad \lambda(\delta_-)\varphi_i(\delta_-) = \lambda(\delta_+)\varphi_i(\delta_+) \tag{11}$$

for special β_i and γ_i only, solving the auxiliary systems of two nonlinear equations

$$\beta_i J_0(\gamma_i\omega_i\delta) = J_0(\omega_i\delta/\sqrt{\alpha_*}) \ , \qquad \beta_i\gamma_i J_1(\gamma_i\omega_i\delta) = (\lambda_*/\sqrt{\alpha_*})J_1(\omega_i\delta/\sqrt{\alpha_*}) \ . \tag{12}$$

Since $\beta_i = J_0(\omega_i\delta/\sqrt{\alpha_*})/J_0(\gamma_i\omega_i\delta)$ evidently, for $F(\gamma_i) := \gamma_i J_1(\gamma_i\omega_i\delta) - \zeta_i J_0(\gamma_i\omega_i\delta)$ where $\zeta_i := (\lambda_*/\sqrt{\alpha_*})J_1(\omega_i\delta/\sqrt{\alpha_*})/J_0(\omega_i\delta/\sqrt{\alpha_*})$, instead of (12) it is sufficient to solve (separately)

all equations $F(\gamma_i)=0$, applying classical Newton iterations $\gamma_i \leftarrow \gamma_i - F(\gamma_i)/F^{\times}(\gamma_i)$ where $F^{\times}(\gamma_i) := \omega_i\delta(\gamma_i J_0(\gamma_i\omega_i\delta)+\zeta_i J_1(\gamma_i\omega_i\delta))$ ($^{\times}$ means the derivative with respect to γ_i).

Inserting (10) and (7) into (6), for any $v\in V$ we receive

$$[(v,\varphi_i)_{r0}+\kappa_*(v,\varphi_i)_{r1}]\dot{\psi}_i-\alpha_0[(v,(r\varphi_i)'/r)_{r0}+\lambda_*(v,(r\varphi_i)'/r)_{r1}]\psi_i=0\ . \tag{13}$$

Taking (9) into account, (13) gets the form

$$[(v,\varphi_i)_{r0}+\kappa_*(v,\varphi_i)_{r1}]\dot{\psi}_i+\alpha_0\omega_i^2[\gamma_i^2(v,\varphi_i)_{r0}+\kappa_*(v,\varphi_i)_{r1}]\psi_i=0\ . \tag{14}$$

Simultaneously, applying the Green-Ostrogradskiĭ theorem, (13) yields

$$[(v,\varphi_i)_{r0}+\kappa_*(v,r\varphi_i)_{r1}]\dot{\psi}_i+\alpha_0[(v',\varphi_i)_{r0}+\lambda_*(v',\varphi_i)_{r1}]\psi_i=\alpha_0[(v(\delta_-)\varphi_i(\delta_-)-\lambda_* v(\delta_+)\varphi_i(\delta_+)]\ . \tag{15}$$

In particular, for $v=\varphi_j$ with any $j\in\{1,2,...\}$, comparing (14) and (15), we have

$$(\varphi_j,\varphi_i)_{r0}+\lambda_*(\varphi_j,\varphi_i)_{r1}=\omega_i^2[\gamma_i^2(\varphi_j,\varphi_i)_{r0}+\kappa_*(\varphi_j,\varphi_i)_{r1}]\ .$$

The mutual exchange of indices i and j then results certain quasi-orthogonality condition

$$(\omega_i^2-\omega_j^2)\kappa_*(\varphi_i,\varphi_j)_{r0}+(\omega_i^2\gamma_i^2-\omega_j^2\gamma_j^2)(\varphi_i,\varphi_j)_{r1}=0\ ;$$

in practice $\gamma_i^2\approx\gamma_j^2\approx\kappa_*$ can be considered.

To find all ψ_i contained in (7), we must solve an eigenproblem

$$M_{ji}\dot{\psi}_i+K_{ji}\psi_i=0$$

for $M_{ji}:=(\varphi_j,r\varphi_i)_0+\kappa(\varphi_j,r\varphi_i)_1$, $K_{ji}:=\alpha_0\omega_i^2\left[(\varphi_j,r\varphi_i)_0+\kappa(\varphi_j,r\varphi_i)_1)\right]$ and for a decomposition

$$\psi_i=V_{ip}\exp(-\Lambda_p t)C_p\ ,$$

using the Einstein summation rule for all indices $i,j,p\in\{1,2,...\}$; Λ_p here are eigenvalues, $V_{i1},V_{i2},...$ eigenvectors (in the matrix form we could write $MV\Lambda=KV$ only) and C_p unknown parameters, needed to be set due to our initial condition. The resulting formulae (assuming $i\neq j$) for effective numerical computations (obtained with the support of MAPLE) are

$$D_{ji}:=(1/\sqrt{\alpha_*})\det\begin{bmatrix}J_0(\omega_j\delta/\sqrt{\alpha_*}) & \omega_j J_1(\omega_j\delta/\sqrt{\alpha_*})\\ J_0(\omega_i\delta/\sqrt{\alpha_*}) & \omega_i J_1(\omega_j\delta/\sqrt{\alpha_*})\end{bmatrix},$$

$$M_{ji}=\delta\lambda_* D_{ji}\left(\frac{1}{\gamma_j^2\omega_j^2-\gamma_i^2\omega_i^2}-\frac{1}{\omega_j^2-\omega_i^2}\right),$$

$$K_{ji}=\alpha_0\omega_i^2\delta\lambda_* D_{ji}\left(\frac{\gamma_i^2}{\gamma_j^2\omega_j^2-\gamma_i^2\omega_i^2}-\frac{1}{\omega_j^2-\omega_i^2}\right),$$

$$2M_{ii} = a^2\kappa J_1^2(\omega_i a/\sqrt{\alpha_*}) + \delta^2(1-\kappa_*)J_0^2(\omega_i\delta/\sqrt{\alpha_*}) + \delta^2\kappa_*(\lambda_*/\gamma_i^2 - 1)J_0^2(\omega_i\delta/\sqrt{\alpha_*}) \ ,$$

$$2K_{ii} = \alpha_0\omega_i^2[a^2\kappa J_1^2(\omega_i a/\sqrt{\alpha_*}) + \delta^2(\gamma_i^2 - \kappa_*)J_0^2(\omega_i\delta/\sqrt{\alpha_*}) + \delta^2\kappa_*(\lambda_* - 1)J_0^2(\omega_i\delta/\sqrt{\alpha_*})] \ .$$

The evaluation of all constants C_p then comes from the equation

$$(v, T_0 - T_\sigma)_{r0} + \kappa_*(v, T_0 - T_\sigma)_{r1} = [(v, \varphi_i)_{r0} + \kappa_*(v, \varphi_i)_{r1}]V_{ip}C_p \ ,$$

i. e. $F_0 + F_1 = MVC$, consequently $C = (MV)^{-1}(F_0 + F_1)$, where

$$F_{0j} := -\beta_j\delta/(\gamma_j\omega_j)[(\eta_0 + \bar{\eta}_1)J_1(\gamma_j\omega_j\delta) - 2\eta_0(\gamma_i\omega_i)^2/\delta J_2(\gamma_j\omega_j\delta) + \eta_0 J_3(\gamma_j\omega_j\delta]$$

with $\eta_0 := Q/(2\pi\lambda_0)$, $\eta_1 := Q/(2\pi\lambda_0)$, $\bar{\eta}_1 := \eta_1\ln(a/\delta)$, and

$$F_{1j} := -\eta_1(\ln(a/r), J_0(\omega_j r/\sqrt{\alpha_*}))_{1r} \ ;$$

just the integral(s) F_{1j} cannot be evaluated analytically in a simple way .

Our final aim is, exploiting the same data as in the preceding section, to minimize a function Φ from (4), admitting also $s = 1$ now, of two positive variables λ_* and κ_* (transformed from λ_1 and α_1). Clearly a (sufficiently large) finite number of Bessel functions is considered in (7) in numerical calculations, thus all matrices M and K, vectors F, etc. are finite. However, it is not so easy to perform the minimization procedure because no simple explicit formulae employable in the Newton iterations are available, thus numerical evaluations of approximate first and second derivatives of Φ are necessary. Fortunately, this can be done e. g. with the support of selected functions from the MATLAB toolbox *optim*, although the quadratic convergence like that in the previous section cannot be expected here.

Fig. 4 makes use the same complete experimental data, even including the initial value for $t = 0$, ignored in the previous section, as Fig. 2 (not their reduced set as Fig. 3), but the realistic wire and specimen sizes in the r-direction have been considered and the finite thermal characteristics of the wire have been taken into account. The green dots in the upper graph refer to the experimental temperature values again; the blue curve corresponds to certain expert initial estimate of λ and κ, whereas the remaining curves document the robustness of the suggested algorithm. The lower graph documents the convergence of numerical iterations (λ- and κ-factors there mean the multiplicative factors to the first estimate) during the minimization of Φ from (4); the dotted lines document a non-negligible number of death ends, caused by the inaccessibility of exact derivatives of Φ, unlike those from the previous section. As shown on Fig. 5, for the temperature on the upper graph and for the heat flux in the r-direction on the lower graph, finally for the approximate solutions of (6), only 10 first Bessel functions are able to generate practically acceptable results. The visible numerical oscillation, namely in the case of the heat flux, derived from the derivatives of T with respect to r, correspond to the jump in the heat source in the initial time. However, the flagrant difference between the measured values and the best simulated time-dependence of temperature on the upper part of Fig. 4, refers to probable influence of other physical processes, not included in the model.

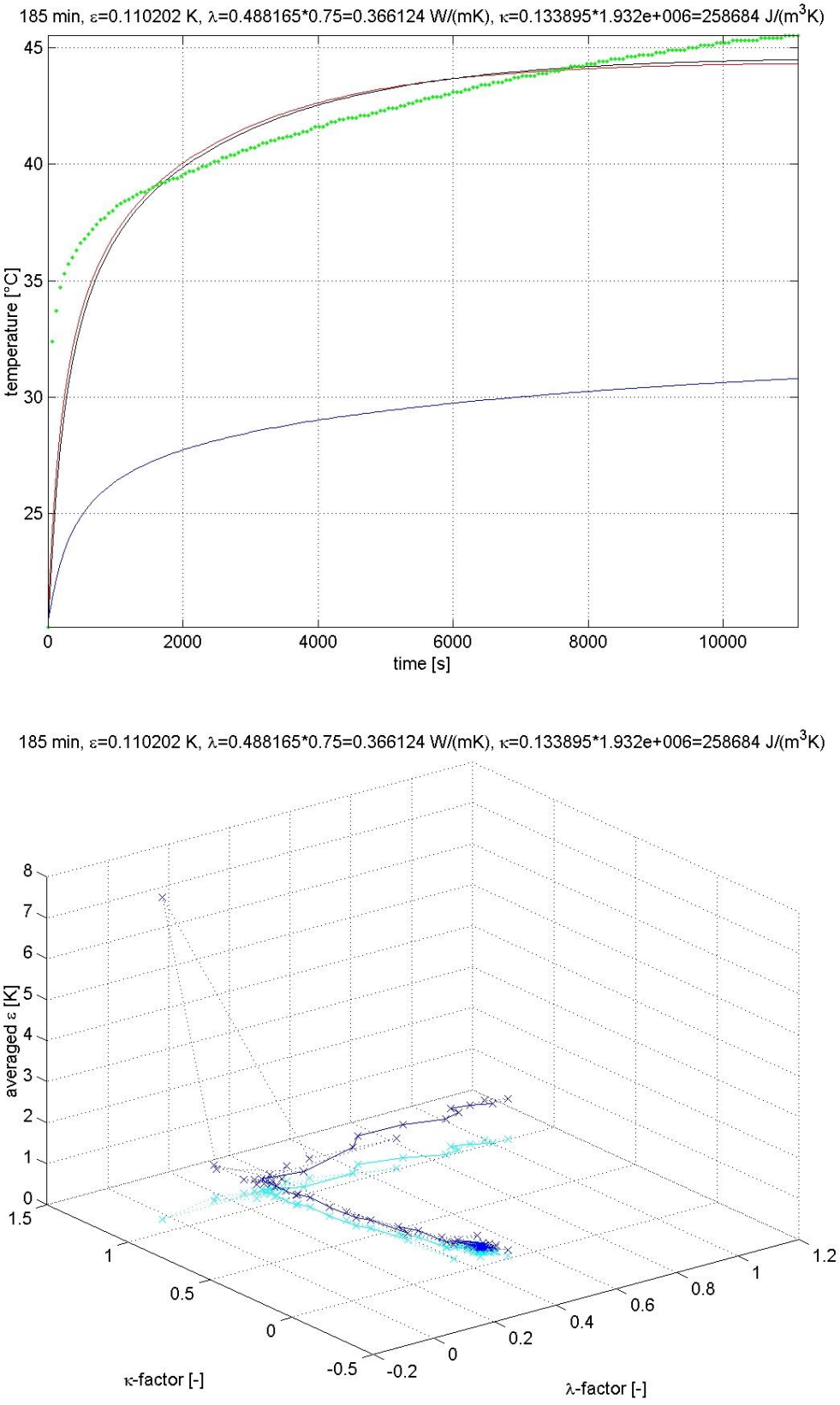

Fig. 4 Development of temperature in time – rough estimate and iterations (upper graph) and corresponding error indicator (lower graph).

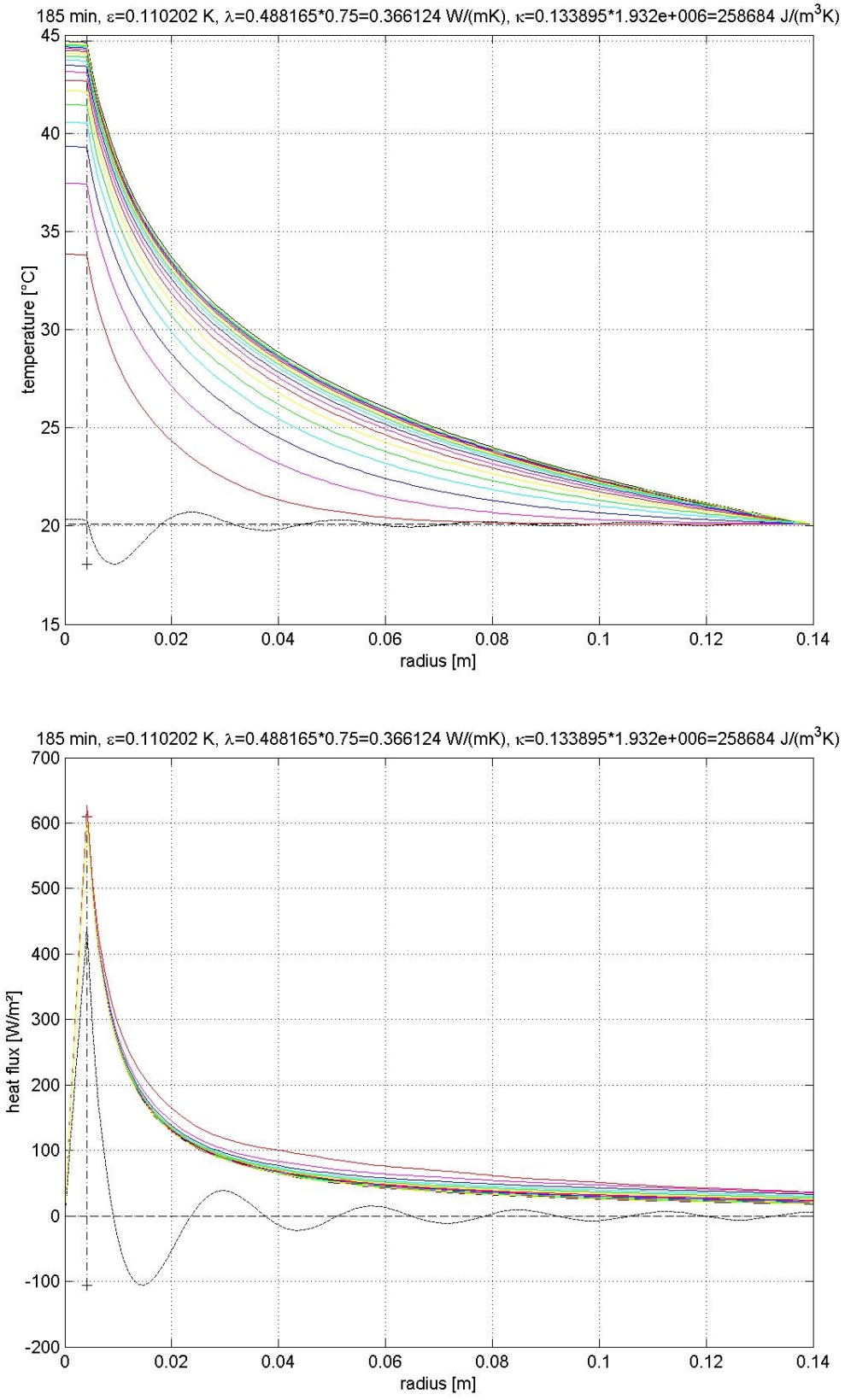

Fig. 5 Distribution of time-variable temperature (upper graph) and heat flux (lower graph), corresponding to Fig. 4.

MORE GENERAL MODELS AND COMPUTATIONAL TECHNIQUES

In general, without simplifying assumptions from two preceding sections, a material specimen can be considered to occupy a domain Ω in the 3-dimensional Euclidean space R^3, supplied by the Cartesian coordinates $x=(x_1,x_2,x_3)$, whose boundary Γ, as illustrated on Fig. 6, consists of two disjoint parts: Γ_c and Γ_i. The standard notation of Lebesgue, Sobolev, Bochner and other spaces of (abstract) functions, compatible with (Fučík, 1980), (Maz'ya, 1984) and (Roubíček, 2005), will be applied in the following text. We shall assume that the choice of Ω guarantees the validity of standard results from the theory of Sobolev spaces, presented in (Roubíček, 2005), p. 16, as the Sobolev imbedding theorem, the trace theorem, etc.; for much more details (involving perverse domains) see (Maz'ya, 1984), p. 63, 222, etc.

Let us take the time t from a time interval $I=[0,\varpi]$ with certain positive final time ϖ; the dot symbol is reserved for partial derivatives with respect to t. Moreover, let us suppose that all variables are not only functions of t and x, defined on $I\times\Omega$, $I\times\Gamma$, etc., but also functions of parameters θ from the sample space A of elementary events. Such sample space must be supplied by the minimal σ-algebra on A and by certain probability measure P. For the function spaces S introduced on some subsets of R^3, as $L^2(\Omega)$ (applying the 3-dimensional Lebesgue measure) or $L^2(\Gamma)$ (applying the 2-dimensional Hausdorff measure), supplied with their standard norms $\|.\|_S$, we are allowed to consider, following (Zabaras, 2004), the abstract function spaces $L^2(A\times I,S)$, for their arbitrary elements ς equipped with the norms

$$\|\varsigma\|_{L^2(A\times I,S)}=\sqrt{\int_A\int_I\|\varsigma(\theta,t,.)\|_S^2\,\mathrm{d}t\,\mathrm{d}P}.$$

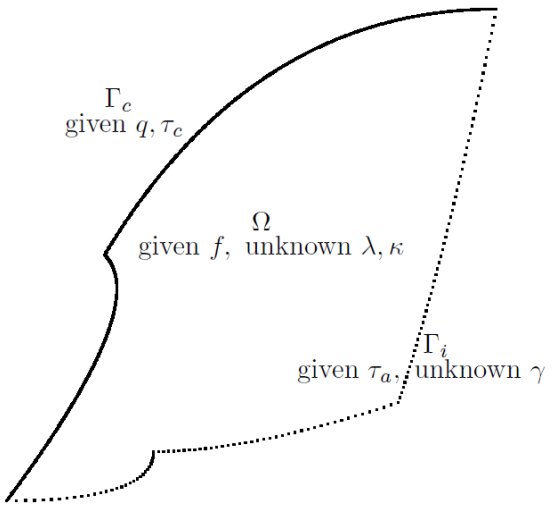

Fig. 6 Geometrical configuration – an illustrative scheme.

Clearly in the deterministic case $A\times I$ degenerates just to I (the integral over A vanishes).

Following two preceding sections, we can also take $T=\tau+T_*$ for an arbitrary temperature level T_*; this can be repeated for all temperature data (without additional comments in the text). Moreover, we shall use the brief notation of scalar products $(.,.)$ in $L^2(A\times I\times\Omega)^d$, $<.,.>_j$ in $L^2(A\times I\times\Gamma_j)$, $(.,.)_t$ in $L^2(A\times\Omega)^d$, $<.,.>_{jt}$ in $L^2(A\times\Gamma_j)$ where d is equal to 1 or

3 (in the sense of Cartesian products), j means i or c and t forces the evaluation at some special time, namely at $t=0$ or $t=\varpi$. The values of the following physical quantities from $L^2(A\times I,S)$ are assumed to be available:

- the body heat flux f for $S=L^{6/5}(\Omega)$,
- the surface heat flux q for $S=L^2(\Gamma_c)$,
- the prescribed temperature τ_c for $S=L^2(\Gamma_c)$,
- the ambient temperature τ_a for $S=L^2(\Gamma_i)$.

At first, let us suppose that $\lambda\in S_\lambda$, $\kappa\in S_\kappa$ and $\gamma\in S_\gamma$ where S_λ and S_κ are admissible sets from $L^\infty(A\times\Omega)$ and S_γ is an admissible set from $L^2(A\times\Gamma_i)$. One can expect that in practical applications $S_\gamma=L^2(A\times\Gamma_i)$ may be considered (boundary heat fluxes depend on some local contact imperfectness, whose detailed analysis is not available), whereas λ and κ belong to some low-dimensional (typically finite) spaces, in the best case of dimension 1 (to obtain constant thermal conductivity and heat capacity and to avoid lack of input data). For simplicity let us also define $\xi=(\lambda,\kappa,\gamma)\in S_\lambda\times S_\kappa\times S_\gamma$.

For arbitrary τ and σ from $L^2(A\times I,W^{1,2}(\Omega))$ (consequently $\tau,\sigma\in L^2(A\times I,L^4(\Gamma))$ and $\tau\sigma\in L^2(A\times I\times\Gamma)$ – see (Roubíček, 2005), p. 17) with $\dot\tau$ and $\dot\sigma$ from $L^2(A\times I\times\Omega)$, choosing some weight $w\in L^2(A\times I\times\Gamma_c)$, let us introduce two functionals

$$\phi(\xi,\tau,\sigma)=(\kappa\dot\tau,\sigma)+(\lambda\nabla\tau,\nabla\sigma)+<\gamma,\tau\sigma>_i-(f,\sigma)-<q,\sigma>_c-<\gamma,\tau_a\sigma>_i,$$

$$G(\tau)=\frac{1}{2}<w,(\tau-\tau_c)^2>_c.$$

To avoid long non-transparent expressions, let us start with the particular case $\xi=(0,0,\gamma)$, introducing $F(\gamma,\tau,\sigma)=\phi(\xi,\tau,\sigma)$. The derivatives of both functionals F and G for $\tilde\tau$ and $\tilde\sigma$ from $L^2(A\times I,W^{1,2}(\Omega))$ and $\tilde\gamma\in S_\gamma$ are

$$DF(\gamma,\tau,\sigma;\tilde\gamma,\tilde\tau,0)=(\kappa\dot{\tilde\tau},\sigma)+(\lambda\nabla\tilde\tau,\nabla\sigma)+<\gamma,\tilde\tau\sigma>_i+<\tilde\gamma,(\tau-\tau_a)\sigma>_i,$$

$$DG(\tau;\tilde\tau)=<w,(\tau-\tau_c)\tilde\tau>_c.$$

Now we are ready to formulate the direct, sensitivity and adjoint problems:

The *direct problem*: Let γ be fixed and $\tau_0=0$. Find such τ that

$$F(\gamma,\tau,\sigma)=0$$

for any σ, i. e.

$$(\kappa\dot\tau,\sigma)+(\lambda\nabla\tau,\nabla\sigma)+<\gamma,(\tau-\tau_a)\sigma>_i=(f,\sigma)+<q,\sigma>_c. \tag{16}$$

The *sensitivity problem*: Let γ and $\tilde\gamma$ be fixed and $\tau_0=0$. Find such $\tilde\tau$ that

$$DF(\gamma,\tau,\sigma;\tilde\gamma,\tilde\tau,0)=0$$

for any σ where τ comes from the direct problem, i. e.

$$(\kappa\dot{\tilde{\tau}},\sigma)+(\lambda\nabla\tilde{\tau},\nabla\sigma)+<\gamma,\tilde{\tau}\sigma>_i=<\tilde{\gamma},(\tau_a-\tau)\sigma>_i \ . \tag{17}$$

The *adjoint problem*: Let γ be fixed and $\sigma_{\varpi}=0$. Find such σ that

$$DF(\gamma,\tau,\sigma;0,\tilde{\tau},0)+DG(\tau;\tilde{\tau})=0$$

for any $\tilde{\tau}$ where τ comes from the direct problem, i. e.

$$-(\kappa\tilde{\tau},\dot{\sigma})-(\lambda\nabla\tilde{\tau},\nabla\sigma)+<\gamma,\tilde{\tau}\sigma>_i=<w,(\tau-\tau_c)\sigma>_c \ . \tag{18}$$

The direct problem does not apparently cover the non-zero values of f_0, q_0 and τ_{a0} correctly. Nevertheless, under the assumption of initial equilibrium (accepted in all experiments)

$$(\kappa\dot{\tau}_0,\sigma)_0+(\lambda\nabla\tau_0,\nabla\sigma)+<\gamma(\tau_0-\tau_{a0}),\sigma>_{i0}=(f_0,\sigma)_0+<q_0,\sigma>_{c0}$$

we can take $f-f_0$, $q-q_0$, $\tau_a-\tau_{a0}$ and $\tau-\tau_0$ instead of f, q, τ_a and τ, which forces the zero initial condition.

With some fixed γ, the overdetermination of (16) is evident. Its solution can be found only in the sense of least squares minimization, i. e. as the minimum of the functional

$$J(\gamma)=G(\tau)=\frac{1}{2}<w,(\tau-\tau_c)^2>_c \tag{19}$$

where τ (dependent on γ) comes from (16). Moreover, taking σ from (18) and $\tilde{\gamma}$ and $\tilde{\tau}$ from (17), we can derive another useful identity

$$<\tilde{\gamma},(\tau_a-\tau)\sigma>_i=<w,(\tau-\tau_c)\tilde{\tau}>_c \ . \tag{20}$$

Applying the Green-Ostrogradskiĭ theorem (on integration by parts), we are also able to convert (16), (17) and (18) (at least in the sense of distributions) into their (more reader-friendly) classical differential forms. Let $\nu=(\nu_1,\nu_2,\nu_3)$ be the local outward unit normal vector on Γ. From (16) we obtain

$$\begin{aligned} \kappa\dot{\tau}-\nabla(\lambda\nabla\tau)+f &= 0 \quad \text{on } A\times I\times\Omega \ , \\ \lambda\nabla\tau\cdot\nu &= q \quad \text{on } A\times I\times\Gamma_c \ , \\ \lambda\nabla\tau\cdot\nu+\gamma(\tau-\tau_a) &= 0 \quad \text{on } A\times I\times\Gamma_i \ , \end{aligned} \tag{21}$$

from (17) similarly

$$\begin{aligned} \kappa\dot{\tilde{\tau}}-\nabla(\lambda\nabla\tilde{\tau}) &= 0 \quad \text{on } A\times I\times\Omega \ , \\ \lambda\nabla\tilde{\tau}\cdot\nu &= 0 \quad \text{on } A\times I\times\Gamma_c \ , \\ \lambda\nabla\tilde{\tau}\cdot\nu+\gamma\tilde{\tau} &= \tilde{\gamma}(\tau_a-\tau) \quad \text{on } A\times I\times\Gamma_i \ , \end{aligned} \tag{22}$$

finally from (18)

$$\begin{aligned} \kappa\dot{\sigma}+\nabla(\lambda\nabla\sigma) &= 0 \quad \text{on } A\times I\times\Omega \ , \\ \lambda\nabla\sigma\cdot\nu &= \beta(\tau-\tau_c) \quad \text{on } A\times I\times\Gamma_c \ , \\ \lambda\nabla\sigma+\gamma\sigma &= 0 \quad \text{on } A\times I\times\Gamma_i \ . \end{aligned} \tag{23}$$

Similar calculations can be repeated both for $\xi=(\lambda,0,0)$ and $\xi=(0,\kappa,0)$, as well as for $\xi=(\lambda,\kappa,0)$: e. g. the right-hand side of (17) obtains the form

$$- (\tilde{\kappa}\dot{\tau},\sigma) - (\tilde{\lambda}\nabla\tau,\nabla\sigma) \ ,$$

consequently such terms occur also in (20). The full (rather long) derivation of corresponding relations for the general case $\xi = (\lambda,\kappa,\gamma)$ is left to the careful reader.

Following the decomposition of $\xi = (\lambda,\kappa,\gamma)$ to $\xi = (0,\kappa,\gamma)$, $\xi = (\lambda,0,\gamma)$ and $\xi = (0,0,\gamma)$, starting from some estimate $\xi^0 = (\lambda^0,\kappa^0,\gamma^0)$, the first step of the computational algorithm should contain (in general) 3 corrections:

- of $\lambda^0 \in S_\lambda$ to $\lambda^1 \in S_\lambda$ with fixed $(\kappa^0,\gamma^0) \in S_\kappa \times S_\gamma$,
- of $\kappa^0 \in S_\kappa$ to $\kappa^1 \in S_\kappa$ with fixed $(\lambda^1,\gamma^0) \in S_\lambda \times S_\gamma$,
- of $\gamma^0 \in S_\gamma$ to $\gamma^1 \in S_\gamma$ with fixed $(\lambda^1,\kappa^1) \in S_\lambda \times S_\kappa$.

The second step repeats the same calculations with ξ^1 (and its components, eventually) instead of ξ^0 and ξ^2 instead of ξ^1, etc. The convergence depends on the kind of corrections.

The available choice for a), b), c) can be the conjugate gradient method for Hilbert spaces, whose basic idea comes from (Axelsson, 1987). Since a), b), especially in the case of low finite dimensions of S_λ and S_κ, can be often handled using the classical Newton algorithm, we shall show such choice in more details for c). Using the upper indices $k \in \{1,2,...\}$ for iterative steps, introducing the simplified notation

$$g^k = (\tau^k(\gamma^k) - \tau_a)\sigma^k \ , \tag{24}$$

motivated by (20), we are able to express the first and second derivatives

$$DJ(\gamma^k;\tilde{\gamma}^k) = <\gamma^k, g^k>_i \ , \qquad D^2J(\gamma^k;\tilde{\gamma}^k;\tilde{\gamma}^k) = <w,(\tilde{\tau}^k(\gamma^k,\tilde{\gamma}^k))^2>_c \ .$$

Consequently the improvement of γ^k can be

$$\gamma^{k+1} = \gamma^k + a^k\tilde{\gamma}^k \tag{25}$$

with the line search parameter

$$a^k = -\frac{DJ(\gamma^k;\tilde{\gamma}^k)}{D^2J(\gamma^k;\tilde{\gamma}^k;\tilde{\gamma}^k)}$$

(generalizing the Newton iteration technique, well-known for a function of one real variable) and that of $\tilde{\gamma}^{k-1}$

$$\tilde{\gamma}^1 = -g^1, \qquad \tilde{\gamma}^k = b^k\tilde{\gamma}^{k-1} - g^k \ \text{ for all } k \in \{2,3,...\} \ , \tag{26}$$

applying the Fletcher-Reeves ratio

$$b^k = \frac{<g^k,g^k>_i}{<g^{k-1},g^{k-1}>_i}$$

or (alternatively) an equivalent evaluation of b^k (e. g. by Hestenes-Stiefel, Polak-Ribiere or Dai-Yuan – cf. (Sun, 2001)).

However, the exact analytical solutions γ^k, $\tilde{\gamma}^k$, τ^k, $\tilde{\tau}^k$, σ^k, etc., from (16), (17), (18), (25) and (26) are (in general) not available; the discretization i) in R^3, ii) in I and iii) in A is needed. The reasonable choice for i) is the finite element technique with Hermitean polynomials as basis functions in an approximation space of some finite integer dimension n (the limit passage $n \to \infty$ is expected again), supporting the direct evaluation of $\nabla\tau$, $\nabla\tilde{\tau}$ and $\nabla\sigma$; for more details and references see (Vala, 2011, 1st item). For ii) this technique can be coupled with the method of discretization in time, based on the construction of the Rothe sequences, using the Crank-Nicholson scheme: for any positive time step h, $m = T/h$ and intervals $I_s^m = \{t \in I : (s-1)h < t \leq sh\}$ with indices $s \in \{1, 2, ...m\}$ we take $(\tau_s^m + \tau_{s-1}^m)/2$ instead of τ and $(\tau_s^m - \tau_{s-1}^m)/h$ instead of $\dot{\tau}$, etc., on I_s^m, with the aim of the limit passage $m \to \infty$.

Some non-classical methods are required for iii), namely the uncertainty representation technique, based on the Karhunen-Loève or polynomial chaos expansions by (Narayanan, 2004) or on the Bayesian approach by (Ma, 2009), compatible with (Ferreira, 2007), p. 25. However, corresponding numerical algorithms may require artificial (non-physical) regularizations, as the implementation of the Tikhonov functional by (Hanke, 1995), Chap. 4, and (Lu, 2010).

Taking into account the scope of the conference, only the basic ideas of all proofs will be sketched here. For the deterministic case the complete existence and convergence analysis should contain the verification:

a) of the existence and uniqueness of a minimizer of (19) (including its generalization containing λ and κ),

b) of the existence and uniqueness of solutions of the time-discretized forms of (16), (17) and (18) and of the convergence of the corresponding Rothe sequences for $m \to \infty$ to original solutions of (16), (17) and (18),

c) of the existence and uniqueness of solutions of the fully discretized forms of (16), (17) and (18), generating for $n \to \infty$ the Rothe sequences from b),

d) on the convergence of the conjugate gradient and other algorithms from the preceding section to a minimizer of (19) from a), including the restarting strategy.

More comments to the items a), b), c), d) follow.

If a weight w is positive (almost) everywhere on Γ_c then the result a) can be derived from the theory of continuous convex functionals by (Fučík, 1980), p. 191. This can be done even independently of the convergence of Rothe sequences by b).

The existence and uniqueness of τ_s^m in b) for particular time steps with $s \in \{1, ..., m\}$ follow from the Lax-Milgram theorem, supported by the standard arguments from the theory of Sobolev spaces (the trace theorem, the Sobolev imbedding theorem, the Cauchy-Schwarz inequality, etc.). Consequently some a priori estimates in appropriate Bochner, Lebesgue and Sobolev spaces, coming from both (continuous and discrete) versions of the Gronwall lemma, guarantee, thanks to the Eberlein-Shmul'yan theorem, the required strong and weak convergences of the Rothe sequences for $m \to \infty$ to τ and $\dot{\tau}$ from (16), to $\tilde{\tau}$ and $\dot{\tilde{\tau}}$ in (17) and to σ and $\dot{\sigma}$ in (18); all preliminaries can be found in (Roubíček, 2005), p. 252.

The full discretization in c) for fixed n and m results in a system of (usually sparse) linear algebraic equations. For a large class of families of decomposition of Ω such systems are regular (even positive definite); consequently the required convergence for $n \to \infty$ follows

from classical results of the finite element approximation theory, e. g. for the Hermitean elements by (Brenner, 2002), p. 75.

The crucial aim of d) is to prove that g^k from (24) tends to zero for $k \to \infty$. This needs the detailed analysis of the iteration steps of the conjugate gradient methods, namely of the evaluation of factors a^k and b^k. Fortunately, thanks to a), the approach of (Sun, 2001) leads to the needed global convergence result (involving slight modifications caused by the improved values of λ and κ).

The technique of proofs in the general stochastic case (where the integrals over A cannot be omitted) is similar to the deterministic one, related to the properties of separable Hilbert spaces and their finite-dimensional approximations. Nevertheless, the standard lemmas and theorems from the (deterministic) variational calculus are valid under more complicated assumptions or are not available at all; this generates some additional problems which has not been closed yet. The stationary (steady-state) problem (where all integrals over I vanish) with $\xi = (0,0,\gamma)$ and $S_\gamma = L^2(A \times \Gamma_i)$ has been analyzed in (Jin, 2008) properly, applying spectral stochastic finite elements and a special Tikhonov regularization, following (Hanke, 1995) and (Lu, 2010). Both approaches to the non-stationary problem (the spectral stochastic and the Bayesian one) with $\xi = (\lambda,\kappa,0)$ with rather general S_λ and S_κ, sketched in (Zabaras, 2004), presented in (Narayanan, 2004) and (Ma, 2009), compatible with (Ferreira, 2007), separately in more details, contain just basic ideas (inciting further non-trivial questions), not the complete set of existence and convergence proofs.

Another needed generalization (in the deterministic case primarily) consists in the more precise physical and mathematical description of the heat transfer (and related) phenomena. To handle corresponding problems properly, most arguments of this section should be generalized substantially because the Lax-Milgram theorem and similar results from the standard variational calculus are not valid outside the linearized formulations like (16), (17), (18) in the integral form, or (20), (21), (22) in the differential one, even for a function $\lambda(\tau)$ instead of λ independent of τ. Namely (20) could obtain the form

$$\begin{aligned} \dot{u} - \nabla \cdot \nabla b(u) + f &= 0 \quad \text{on } A \times I \times \Omega\,, \\ \nabla b(u) \cdot \nu + \gamma_e(u)(|\,a(u)\,|^{N-1}\, a(u) - \tau_a^N) &= q \quad \text{on } A \times I \times \Gamma_c\,, \\ \nabla b(u) \cdot \nu + \gamma_e(u)(|\,a(u)\,|^{N-1}\, a(u) - \tau_a^N) + \gamma(u)(b(u) - \tau_a) &= 0 \quad \text{on } A \times I \times \Gamma_i\,; \end{aligned} \tag{24}$$

here $\tau = c^{-1}(u)$, $b(u) := \lambda(\tau)$ and λ means a function whose derivative is λ. Moreover, new material characteristics $a(u)$ and $\gamma_e(u)$, coming from the analysis of thermal radiation, together with a constant exponent N from the Stefan-Boltzmann law, occur in the nonlinear system of partial differential equations of evolution (24). The proper mathematical study of the system (24) is not easy, being conditioned by assumed growth properties of all above sketched functions: in some special cases we can study its weak solvability, using test functions from V, introduced now as the space of all functions from $W^{1,2}(\Omega)$ whose traces belong to $L^N(\Gamma)$, in more general cases we must take up with its very weak solvability, following (Roubíček, 2005), p. 241, applying test functions from $W^{1,\infty,\infty}(I, W^{1,\infty}(\Omega), L^{6/5}(\Omega))$. The more detailed analysis of this type, including some stochastic considerations, has been prepared for (Vala, 2013).

CONCLUSIONS AND APPLICATIONS

The problem of heat transfer in building materials, seemingly a simple linear problem of evolution of temperature, as a special case of the conservation of energy from classical thermodynamics, brings substantial difficulties in its inverse (in general probabilistic) formulation, even for very special geometrical configurations, as the hot wire one, discussed in this paper. Moreover, under hard experimental conditions, namely at high temperatures, the temperature-dependence of material characteristics and the influence of other physical processes cannot be neglected. This generates still open problems both in experimental research and in computational simulations, including their physical and mathematical background, motivated the research activities for the near future.

One practical application is to the production on dense-shaped refractory products, in the cooperation of BUT with PD-Refractories in Velké Opatovice (former Moravian Fire and Schistous Clay Plants, Czech Republic). The hot-wire experiments can be organized in the similar way as on Fig. 1, although the illustrative experimental and simulation results, presented and discussed in this paper, have been derived for the standard environment temperature where such materials as constantan (a copper-nickel alloy), substituted the final use of (much more expensive) platinum-like ones.

Another important application is connected with the material design for the high-temperature thermal storage, as one part of the large Swedish-Czech research project of the efficient exploitation of solar energy using optical fibers. This project for the time period 2012–14 is supported by the Technology Agency of the Czech Republic (Reg. No. 02021231). Fig. 7 shows a small model of the thermal storage, whose complex design is a substantial part of this project; all technical details (because of patent protection) are not available to publication yet.

Fig. 7 Model of the thermal storage – one part of the proposed system of effective exploitation of sun energy using optical fibres.

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